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Geometric Convergence of Chebyshev Rational Approximations on $[0, +\infty)$

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1. INTRODUCTION

Let f be a continuous real-valued function on $[0, +\infty)$ and define

$$\|f\|_r = \sup\{|f(x)|: 0 \leqslant x \leqslant r\} \qquad \text{for} \quad r > 0$$

and

$$\|f\|_\infty = \sup\{|f(x)|: 0 \leqslant x\}.$$

For each nonnegative integer n define π_n to be the set of algebraic polynomials with real coefficients of degree not exceeding n.

We investigate the following problem: For which functions $f \in C[0, +\infty)$ does there exist a number $q > 1$ and a sequence of polynomials $\{p_n\}_{n=0}^\infty$ such that $p_n \in \pi_n$, $n = 0, 1, 2,\ldots$, and

$$\limsup_{n\to\infty} (\|(1/f) - (1/p_n)\|_\infty)^{1/n} \leqslant 1/q? \tag{1.1}$$

The complete answer to this problem is not yet known although many authors in recent years have investigated this. If there exists a $q > 1$ and a sequence of polynomials such that this happens for some f then we say *f has geometric convergence*. Thus we seek to classify all $f \in C[0, +\infty)$ which have geometric convergence.

The first result on this problem established that $f(x) = e^x$ has geometric convergence (see [4]). This result was extended to other functions in [6]. The first in-depth study was made by Meinardus *et al.* [7]. They obtained a necessary condition as well as a sufficient condition for f to have geometric convergence. Since the appearance of [7], several researchers have suggested

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that the necessary condition obtained in [7] may also be sufficient. In particular, Roulier and Taylor [9] obtain a less restrictive sufficient condition and conjecture that the necessary condition in [7] is also sufficient. Blatt [1, 2] further weakens the hypotheses of the sufficient condition given in [7]. These results further suggested that the necessary condition in [7] might also be sufficient.

In this paper we obtain a new necessary condition for geometric convergence. This in turn, provides the machinery for constructing a counterexample to the above conjecture. That is, a function f is defined which fails to have geometric convergence, and yet f satisfies the necessary condition obtained in [7]. In addition we obtain a new sufficient condition that is essentially different from those already known. This, in turn, will be used to generate new examples of functions which have geometric convergence but with properties that are somewhat surprising.

The details of the previous results and the terminology needed to understand these are presented in the next section.

Other related results and a large bibliography of such results appear in the survey paper [8].

2. Notation and Previous Results

Let $r > 0$ and $s > 1$ be given, and let $E(r, s)$ denote the unique ellipse in the complex plane with foci at $x = 0$ and $x = r$ and semimajor and semiminor axes a and b, respectively, with $b/a = (s^2 - 1)/(s^2 + 1)$. If $f(z)$ is any function analytic inside and on the boundary of $E(r, s)$ define

$$M_f(r, s) = \max\{|f(z)| : z \in E(r, s)\}.$$

The necessary condition obtained in [7] is given in the following theorem.

Theorem 2.1. *Let f be a real continuous function ($\not\equiv 0$) on $[0, +\infty)$, and assume that there exist a sequence of real polynomials $\{p_n\}_{n=0}^{\infty}$, with $p_n \in \pi_n$ for $n = 0, 1, \ldots,$ and $q > 1$ such that*

$$\limsup_{n \to \infty} (\|(1/f) - (1/p_n)\|_\infty)^{1/n} = 1/q < 1. \tag{2.1}$$

Then, there exists an entire function $F(z)$ with $F(x) = f(x)$ for all $x \geq 0$, and F is of finite order ρ. In addition, for every $s > 1$, there exist constants $K = K(s, q) > 0$, $\theta = \theta(s, q) > 0$, and $r_0 = r_0(s, q) > 0$ such that

$$M_F(r, s) \leq K(\|f\|_r)^\theta \qquad \textit{for all} \quad r \geq r_0. \tag{2.2}$$

The sufficient condition in [7] is given by the following theorem.

THEOREM 2.2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be an entire function with $a_0 > 0$ and $a_k \geqslant 0$ for all $k \geqslant 1$. If there exists real numbers $A > 0$, $s > 1$, $\theta > 0$, and $r_0 > 0$ such that*

$$M_f(r, s) \leqslant A(\|f\|_r)^{\theta} \qquad \text{for all} \quad r \geqslant r_0$$

then there exists a sequence of real polynomials $\{p_n\}_{n=0}^{\infty}$ with $p_n \in \pi_n$ for each $n \geqslant 0$, and a real number $q \geqslant s^{1/(1+\theta)} > 1$ such that

$$\limsup_{n\to\infty} (\|(1/f) - (1/p_n)\|_{\infty})^{1/n} = 1/q < 1.$$

It is suggested in [7] that the hypotheses here are probably too strong. A more general theorem was given by Roulier and Taylor [9] and this was generalized further by Blatt in [1, 2]. We give this latter theorem [2]. In order to do this, we need to introduce some additional notation. Let

$$0 \leqslant x_1 < x_2 < \cdots < x_L < \infty$$

with corresponding nonnegative integers $\beta_1, \ldots, \beta_L$ be given. Define

$$N = \left\{ h(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \;\middle|\; \begin{array}{l} a_\nu \text{ real, } h \text{ an entire transcendental function whose} \\ \text{zeros in } [0, +\infty) \text{ are precisely at } x_i \text{ with order } \beta_i \\ i = 1, \ldots, L, \text{ and } \lim_{x\to+\infty} h(x) = +\infty \end{array} \right\}$$

and

$$\tilde{N} = \left\{ h(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu \;\middle|\; \begin{array}{l} a_\nu \text{ real, } h \text{ entire, } h \neq 0, h \text{ has zeros at } x_i \text{ with order} \\ \geqslant \beta_i \; i = 1, \ldots, L. \end{array} \right\}$$

THEOREM 2.3. *Let $f \in N$ and assume that for every $s > 1$, there exist constants $K(s) > 0$, $\theta(s) > 0$, and $r(s) > 0$ such that*

$$M_f(r, s) \leqslant K(s)(\|f\|_r)^{\theta(s)} \qquad \text{for all} \quad r \geqslant r(s). \tag{2.4}$$

Further, assume that there exist entire functions $f_1, f_2 \in \tilde{N}$ such that

$$f = f_1 + f_2, \tag{2.5}$$

f_1 has geometric convergence and there exists a real number $r_0 > 0$ such that f_1 is nondecreasing for $r \geqslant r_0$, (2.6)

there exists $B > 0$ such that $f_2(x) \geqslant -B$ for all $x \geqslant 0$, (2.7)

there exist $\psi > 0$ and $A > 0$ such that $f_2(x) \leqslant A[f_1(x)]^{\psi}$ for all $x \geqslant r_0$, (2.8)

there exists a sequence of positive integers $\{n_j\}$ for which $1 < n_{j+1}/n_j < \rho$ (ρ a fixed real number) and $f_2^{(n_j+1)}(x) \leqslant 0$ for all $x \geqslant 0$, $j = 1, 2, \ldots$. (2.9)

Then f has geometric convergence.

3. A New Necessary Condition and a Counterexample

In the following theorem we show that if f has geometric convergence, then f cannot oscillate too badly as x approaches $+\infty$.

Theorem 3.1. *Let $f \in C[0, +\infty)$ satisfy*

$$\lim_{x \to +\infty} f(x) = +\infty, \tag{3.1}$$

and let $\{x_j\}_{j=0}^{\infty}$ be a sequence of real numbers such that

$$0 < x_0 < x_1 < \cdots, \tag{3.2}$$

$$\lim_{j \to +\infty} x_j = +\infty, \tag{3.3}$$

$$f(x_{2j}) \leq f(x_{2j+2}) \quad \textit{for} \quad j = 0, 1, 2, \ldots, \tag{3.4}$$

$$f(x_{2j-1}) \leq f(x_{2j+1}) \quad \textit{for} \quad j = 1, 2, \ldots, \tag{3.5}$$

$$f(x_{2j}) < f(x_{2j-1}) \quad \textit{for} \quad j = 1, 2, \ldots, \tag{3.6}$$

$$\lim_{j \to \infty} \frac{f(x_{2j})}{f(x_{2j-1})} = 0 \tag{3.7}$$

and

$$\textit{for any} \quad r > 1 \qquad \lim_{j \to \infty} \frac{f(x_{2j})}{r^j} = 0. \tag{3.8}$$

Then f does not have geometric convergence.

Proof. Assume that f does have geometric convergence. It follows from (1.1) that there exist $q > 1$, sequence $\{p_n\}_{n=0}^{\infty}$ with $p_n \in \pi_n$, $n = 0, 1, \ldots$, and $N_0 > 0$ such that $n \geq N_0$ implies

$$\|(1/f) - (1/p_n)\|_\infty \leq 1/q^n. \tag{3.9}$$

Now (3.1) implies the existence of $r_0 > 0$ such that

$$f(x) \geq 1 \qquad \text{for} \quad x \geq r_0. \tag{3.10}$$

The combination of (3.3) and (3.10) gives the existence of $J_0 > 0$ such that

$$j \geq J_0 \text{ implies } f(x_j) \geq 1. \tag{3.11}$$

Furthermore, the combination of (3.4), (3.7), and (3.11) gives

$$\lim_{j \to \infty} \frac{f(x_{2j-2})}{f(x_{2j-1})} = 0. \tag{3.12}$$

Choose $J_1 \geqslant J_0$ ((3.7) and (3.12)) so that $j \geqslant J_1$ implies

$$\frac{f(x_{2j-2})}{f(x_{2j-1})} \leqslant \frac{1}{2} \quad \text{and} \quad \frac{f(x_{2j})}{f(x_{2j-1})} \leqslant \frac{1}{2}. \tag{3.13}$$

We may now use (3.11) and (3.13) to obtain for $j \geqslant J_1$

$$\begin{aligned} &\frac{1}{f(x_{2j-2})} - \frac{1}{f(x_{2j-1})} = \frac{1}{f(x_{2j-2})}\left(1 - \frac{f(x_{2j-2})}{f(x_{2j-1})}\right) \geqslant \frac{1}{2f(x_{2j-2})} \\ \text{and} \quad &\frac{1}{f(x_{2j})} - \frac{1}{f(x_{2j-1})} = \frac{1}{f(x_{2j})}\left(1 - \frac{f(x_{2j})}{f(x_{2j-1})}\right) \geqslant \frac{1}{2f(x_{2j})}. \end{aligned} \tag{3.14}$$

It follows from (3.8) that there exists integer $J_2 \geqslant J_1$ such that $j \geqslant J_2$ implies both

$$\frac{f(x_{2j})}{q^j} \leqslant \frac{1}{8} \quad \text{and} \quad \frac{f(x_{2j-2})}{q^j} \leqslant \frac{1}{8}. \tag{3.15}$$

Note, furthermore, that if $k \geqslant j \geqslant J_2$ then from (3.15) we have

$$\frac{f(x_{2j})}{q^k} \leqslant \frac{1}{8} \quad \text{and} \quad \frac{f(x_{2j-2})}{q^k} \leqslant \frac{1}{8}. \tag{3.16}$$

We now use (3.9), (3.14), and (3.16) to observe that for $n \geqslant N_0$ and

$$J_2 \leqslant j \leqslant n$$

$$\begin{aligned} &\frac{1}{p_n(x_{2j-2})} - \frac{1}{p_n(x_{2j-1})} \\ &\quad = \frac{1}{f(x_{2j-2})} - \frac{1}{f(x_{2j-1})} + \frac{1}{p_n(x_{2j-2})} - \frac{1}{f(x_{2j-2})} + \frac{1}{f(x_{2j-1})} - \frac{1}{p_n(x_{2j-1})} \\ &\quad \geqslant \frac{1}{2f(x_{2j-2})} - \frac{2}{q^n} \\ &\quad = \frac{1}{2f(x_{2j-2})}\left(1 - \frac{4f(x_{2j-2})}{q^n}\right) \\ &\quad \geqslant \frac{1}{4f(x_{2j-2})}. \end{aligned}$$

That is, if $J_3 \geqslant \max(J_2, N_0)$ then $J_3 \leqslant j \leqslant n$ implies

$$\frac{1}{p_n(x_{2j-2})} - \frac{1}{p_n(x_{2j-1})} \geqslant \frac{1}{4f(x_{2j-2})}. \tag{3.17}$$

In a similar fashion we can show that $J_3 \leqslant j \leqslant n$ implies

$$\frac{1}{p_n(x_{2j})} - \frac{1}{p_n(x_{2j-1})} \geqslant \frac{1}{4f(x_{2j})}. \tag{3.18}$$

It follows from (3.9) and (3.10) that $p_n(x) \neq 0$ if $x \geqslant r_0$ and $n \geqslant N_0$. It now follows from this, (3.17) and (3.18) that $1/p_n$ has a relative minimum on each of the intervals

$$(x_{2j-2}, x_{2j}), \qquad J_3 \leqslant j \leqslant n$$

and a relative maximum on each of the intervals

$$(x_{2j-1}, x_{2j+1}), \qquad J_3 \leqslant j \leqslant n.$$

Thus $1/p_n$ has at least $2(n - J_3 + 1)$ relative extrema on $[r_0, +\infty)$. But if we fix J_3 and take n large enough we see that $1/p_n$ must have at least n relative extrema on $[r_0, +\infty)$. But this implies that $p_n'(x) = 0$ for at least n distinct points. Hence, $p_n' \equiv 0$ and p_n is a constant for n sufficiently large. This is a contradiction since f is not a constant. ∎

We now use Theorem 3.1 to construct a function f which satisfies the necessary conditions obtained in Theorem 2.1 but which fails to have geometric convergence.

EXAMPLE 3.1. Define the entire function

$$F(z) = z + 1 + e^z \sin^2 z;$$

and let f be the restriction of F to the real line;

$$f(x) = x + 1 + e^x \sin^2 x.$$

We will show that f satisfies both the conclusion of Theorem 2.1 and the hypotheses of Theorem 3.1. Thus f will be the counterexample alluded to in Section 1.

Define the sequence $\{x_j\}_{j=0}^{\infty}$ by

$$x_j = j\pi/2, \qquad j = 0, 1, 2,\ldots.$$

Then we have

$$\begin{aligned} f(x_j) &= x_j + 1 && \text{for } j \text{ even,} \\ &= x_j + 1 + e^{x_j} && \text{for } j \text{ odd.} \end{aligned}$$

Clearly, this f and the sequence $\{x_j\}_{j=0}^{\infty}$ satisfy (3.1)–(3.8). Thus f does not have geometric convergence.

Given $s > 1$ define

$$\mu = \tfrac{1}{2}[1 + \tfrac{1}{2}(s + (1/s))].$$

It is easy to show that

$$M_F(r, s) \leqslant 2(\|f\|_r)^{6\mu} \qquad \text{for} \quad r \geqslant 2\pi.$$

Moreover, F is of finite order. Hence, f satisfies the conclusion of Theorem 2.1 but does not have geometric convergence.

4. A New Sufficient Condition

The following theorem gives a sufficient condition for a function f to have geometric convergence. It is essentially different from the results of Roulier and Taylor [9] and of Blatt [1, 2]. In order to demonstrate this, an example based on this theorem is given; the example is not obtainable from any of the previously published results.

THEOREM 4.1. *Let $f \in C[0, +\infty)$ satisfy*

$$f(x) \geqslant \eta > 0 \qquad on\ [0, +\infty), \tag{4.1}$$

$$\lim_{x\to+\infty} f(x) = +\infty, \tag{4.2}$$

there exist real-valued functions h and g such that h and g are restrictions of entire functions and $f'(x) = h^2(x) + g^2(x)$, (4.3)

there exist numbers $A > 0$, $s > 1$, $\theta > 0$ and $r_0 > 0$ such that

$$M_{h^2}(r, s) + M_{g^2}(r, s) \leqslant A(\|f\|_r)^\theta \qquad for \quad r \geqslant r_0 . \tag{4.4}$$

Then f has geometric convergence, and the q in (1.1) *satisfies*

$$q \geqslant s^{1/2(2+\theta)} > 1.$$

The proof of this theorem requires three preliminary lemmas.

LEMMA 1. *Let f, h, and g be as in hypotheses* (4.1), (4.2), *and* (4.3) *of Theorem* 4.1. *Then for each fixed $r > 0$ there is a sequence of polynomials $\{p_{2n}\}_{n=0}^{\infty}$ with $p_{2n} \in \pi_{2n}$, $n = 0, 1,\ldots$ and for which*

$$p_{2n}(x) > 0 \qquad for\ all\ real\ x\ and\ n = 0, 1,\ldots, \tag{4.5}$$

and for each $s > 1$

$$\|f' - p_{2n}\|_r \leqslant \frac{8}{(s-1)\,s^n}\,[M_{h^2}(r, s) + M_{g^2}(r, s)]. \tag{4.6}$$

Proof. For each nonnegative integer n let u_n be the polynomial of best approximation from π_n on $[0, r]$ to g, and let v_n be the polynomial of best approximation from π_n on $[0, r]$ to h. Define $E_n(g)$ and $E_n(h)$ by

$$E_n(g) = \|g - u_n\|_r \quad \text{for} \quad n = 0, 1,...$$
and
$$E_n(h) = \|h - v_n\|_r \quad \text{for} \quad n = 0, 1,.... \tag{4.7}$$

Define

$$p_{2n}(x) = u_n^2(x) + v_n^2(x) + E_{2n}(f') \quad \text{for} \quad n = 0, 1,...., \tag{4.8}$$

where $E_{2n}(f')$ is the degree of best uniform approximation to f' on $[0, r]$ by polynomials from π_{2n}. Then we have

$$\begin{aligned} f'(x) - p_{2n}(x) &= g^2(x) + h^2(x) - u_n^2(x) - v_n^2(x) - E_{2n}(f') \\ &= (g(x) - u_n(x))(g(x) + u_n(x)) \\ &\quad + (h(x) - v_n(x))(h(x) + v_n(x)) - E_{2n}(f'). \end{aligned}$$

Thus for each $n = 0, 1,...$

$$\begin{aligned} \|f' - p_{2n}\|_r \leqslant \|g - u_n\|_r \cdot \|g + u_n\|_r + \|h - v_n\|_r \\ \cdot \|h + v_n\|_r + E_{2n}(f'). \end{aligned} \tag{4.9}$$

But it is well known that for each $n = 0, 1,....$ we have

$$\|g + u_n\|_r \leqslant 3\|g\|_r \leqslant 3M_g(r, s)$$
and
$$\|h + v_n\|_r \leqslant 3\|h\|_r \leqslant 3M_h(r, s). \tag{4.10}$$

Moreover, by a theorem of S.N. Bernstein [5, p. 91] we have, for any $s > 1$, $n = 0, 1,...$

$$E_n(g) \leqslant \frac{2M_g(r, s)}{(s - 1)\, s^n},$$

$$E_n(h) \leqslant \frac{2M_h(r, s)}{(s - 1)\, s^n}, \tag{4.11}$$

and

$$E_{2n}(f') \leqslant \frac{2M_{f'}(r, s)}{(s - 1)\, s^{2n}} < \frac{2}{(s - 1)\, s^n} [M_{g^2}(r, s) + M_{h^2}(r, s)].$$

A combination of (4.9), (4.10), and (4.11) together with the observation that

$$(M_g(r, s))^2 = M_{g^2}(r, s)$$

and

$$(M_h(r, s))^2 = M_{h^2}(r, s),$$

gives

$$\|f' - p_{2n}\|_r \leqslant \frac{8}{(s-1)\,s^n}\,[M_{g^2}(r, s) + M_{h^2}(r, s)].$$

This completes the proof of Lemma 1.

LEMMA 2. *For each $n = 0, 1,\ldots$ define the sets*

$$\pi_n^+ = \{p \in \pi_n \mid p(x) \geqslant 0 \textit{ for all } x\}$$

and

$$\pi_n' = \{p \in \pi_n \mid p'(x) \geqslant 0 \textit{ for all } x\}.$$

If $r > 0$ is given and if $f \in C[0, r]$ define

$$E_{n,r}^+(f) = \inf_{p \in \pi_n^+} \|f - p\|_r$$

and

$$E_{n,r}'(f) = \inf_{p \in \pi_n'} \|f - p\|_r .$$

Then if f has a continuous derivative on $[0, r]$ we have

$$E_{n+1,r}'(f) \leqslant rE_{n,r}^+(f') \qquad \textit{for} \quad n = 0, 1,\ldots . \tag{4.12}$$

The proof is a direct application of classical techniques and is consequently omitted.

LEMMA 3. *Let f and g be real-valued functions which are restrictions of entire functions and which satisfy: there are constants $A > 0$, $\theta > 0$, $s > 1$, $r_0 > 0$ for which*

$$M_g(r, s) \leqslant Af(r)^\theta \ \textit{ for } r \geqslant r_0 . \tag{4.13}$$

Then either g is a polynomial or given any positive integer M there is a number $R_M > 0$ such that

$$f(r) \geqslant r^M \qquad \textit{for} \quad r \geqslant R_M . \tag{4.14}$$

The proof of this lemma is an easy application of Liouville's theorem. The details are omitted.

Proof of Theorem 4.1. The method of proof is essentially the same as that used in proving the sufficient condition for geometric convergence in Theorem 2.2. We may assume that f is not a polynomial, since the theorem is trivial in this case.

For each $r > 0$ define $\{q_n(x, r)\}_{n=0}^{\infty}$ with $q_n \in \pi_n'$ so that

$$\|f - q_n(\cdot, r)\|_r = E_{n,r}'(f).$$

We know that this sequence of best restricted approximations exists for each $r > 0$. Now for each r define

$$p_n(x, r) = q_n(x, r) + E'_{n,r}(f), \qquad n = 0, 1, \ldots .$$

This guarantees that

$$\begin{aligned} &p_n(x, r) \geqslant f(x) \geqslant \eta > 0 \qquad \text{for all} \quad x \text{ in } [0, r], \\ \text{and} \quad &p_n(x, r) \geqslant f(r) \geqslant \eta > 0 \qquad \text{for all} \quad x \geqslant r \text{ for } n = 0, 1, \ldots . \end{aligned} \tag{4.15}$$

Moreover,

$$\| f - p_n(\cdot, r) \|_r \leqslant 2E'_{n,r}(f), \qquad n = 0, 1, \ldots . \tag{4.16}$$

Now the fact that f is positive and increasing together with (4.15) gives

$$\left| \frac{1}{f(x)} - \frac{1}{p_n(x, r)} \right| \leqslant \frac{2}{f(r)} \qquad \text{for} \quad x \geqslant r, \quad n = 0, 1, \ldots . \tag{4.17}$$

Also, (4.16) and (4.15) give

$$\left| \frac{1}{f(x)} - \frac{1}{p_n(x, r)} \right| = \frac{| p_n(x, r) - f(x) |}{f(x)\, p_n(x, r)} \leqslant \frac{2E'_{n,r}(f)}{\eta^2}$$
$$\text{for} \quad 0 \leqslant x \leqslant r, \quad n = 0, 1, \ldots . \tag{4.18}$$

But Lemmas 1 and 2 combine to give

$$\begin{aligned} E'_{2n+1,r}(f) &= \| f - q_{2n+1}(\cdot, r) \|_r \leqslant rE^{+}_{2n,r}(f') \\ &\leqslant \frac{8r}{(s-1)\, s^n} [M_{g^2}(r, s) + M_{h^2}(r, s)]. \end{aligned} \tag{4.19}$$

Now let A, s, θ, r_0 be such that (4.4) holds and combine this with (4.18) and (4.19) to obtain

$$\begin{aligned} \left\| \frac{1}{f} - \frac{1}{p_{2n+1}(\cdot, r)} \right\|_r &\leqslant \frac{16r}{\eta^2(s-1)\, s^n} [M_{g^2}(r, s) + M_{h^2}(r, s)] \\ &\leqslant \frac{16rA}{\eta^2(s-1)\, s^n} \| f \|_r^\theta \qquad \text{for} \quad r \geqslant r_0 . \end{aligned}$$

But since f is positive and increasing on $[0, +\infty)$, this inequality implies that

$$\begin{aligned} \left\| \frac{1}{f} - \frac{1}{p_{2n+1}(\cdot, r)} \right\|_r &\leqslant \frac{16A}{\eta^2(s-1)\, s^n} f(r)^\theta\, r \\ &= \frac{Brf(r)^\theta}{s^n} \qquad \text{for} \quad r \geqslant r_0 , \end{aligned} \tag{4.20}$$

where $B = 16A/\eta^2(s-1)$ does not depend on r.

Recall that we are working under the assumption that f is not a polynomial. We now combine (4.1), (4.3), and (4.4) to obtain

$$M_{f'}(r, s) \leqslant Af(r)^{\theta} \qquad \text{for} \quad r \geqslant r_0 .$$

But f' is not a polynomial. Thus an application of Lemma 3 gives $r_1 \geqslant r_0$ for which $r \geqslant r_1$ implies

$$f(r) \geqslant r.$$

Thus for $r \geqslant r_1$ (4.20) becomes

$$\left\| \frac{1}{f} - \frac{1}{p_{2n+1}(\cdot, r)} \right\|_r \leqslant \frac{Bf(r)^{\psi}}{s^n} \qquad \text{for} \quad r \geqslant r_1 \tag{4.21}$$

where $\psi = \theta + 1$.

Now since $\lim_{r \to \infty} f(r) = +\infty$ we have $N > 0$ and $r(n) \geqslant r_1$ for each $n \geqslant N$ such that

$$f(r(n)) = s^{n/(1+\psi)}.$$

Note that $\lim_{n \to \infty} r(n) = +\infty$. Now for each $n \geqslant N$ set

$$p_{2n+1}(x) = p_{2n+1}(x, r(n)).$$

Then from (4.21) we have

$$\left\| \frac{1}{f} - \frac{1}{p_{2n+1}} \right\|_{r(n)} \leqslant \frac{Bf(r(n))^{\psi}}{s^n} = \frac{Bs^{n\psi/(1+\psi)}}{s^n} = \frac{B}{s^{n/(1+\psi)}} \qquad \text{for} \quad n \geqslant N, \tag{4.22}$$

and from (4.17) we have

$$\left| \frac{1}{f(x)} - \frac{1}{p_{2n+1}(x)} \right| \leqslant \frac{2}{f(r(n))} = \frac{2}{s^{n/(1+\psi)}} \qquad \text{for} \quad x \geqslant r(n) \quad \text{and} \quad n \geqslant N. \tag{4.23}$$

A combination of (4.22) and (4.23) now gives

$$\left\| \frac{1}{f} - \frac{1}{p_{2n+1}} \right\|_{\infty} \leqslant \frac{B + 2}{s^{n/(1+\psi)}} \qquad \text{for} \quad n \geqslant N. \tag{4.24}$$

The proof is now completed by setting $p_n(x) = 1$ for $n < 2N + 1$ and if $n \geqslant 2N + 1$ set

$$p_n = p_{2k+1} \qquad \text{if} \quad n = 2k + 1 \text{ or } 2k + 2. \quad \blacksquare$$

We now employ Theorem 4.1 in conjunction with Theorem 2.3 to obtain an example of a function f with geometric convergence which is not obtainable from the previous sufficient conditions.

EXAMPLE 4.1. Let

$$f_1(x) = \tfrac{1}{8}e^{2x}[2 + \sin(2x) + \cos(2x)]$$

and let

$$f_2(x) = e^{-x}.$$

Let $f(x) = f_1(x) + f_2(x)$. Note that

$$f_1'(x) = (e^x \cos x)^2$$

and

$$f_2'(x) = -e^{-x}.$$

It is easy to see that f_1 satisfies the hypotheses of Theorem 4.1 with

$$h(x) = e^x \cos x \qquad \text{and} \qquad g(x) = 0.$$

Hence, f_1 has geometric convergence.

It is also easy to see that $f = f_1 + f_2$ satisfies the conditions of Theorem 2.3. Hence, f has geometric convergence.

Notice, however, that

$$f'(x) = (e^x \cos x)^2 - e^{-x}$$

will assume negative values for arbitrarily large x. Thus there is no r for which f is increasing on $[r, +\infty)$. Functions with behavior similar to that of the f in Example 4.1 ($\lim_{x\to+\infty} f(x) = +\infty$, f not increasing on $[r, +\infty)$ for any r, and f has geometric convergence) are not readily obtained from any combination of theorems found in the literature prior to Theorem 4.1 of this paper.

We end this section with a corollary to Theorem 4.1 which shows that Theorem 4.1 is closely related to an approach suggested in a private communication to the second author by Professor G. D. Taylor.

COROLLARY. *Suppose that f is a positive real-valued function on $[0, +\infty)$ and is the restriction of an entire function, and that $\lim_{x\to\infty} f(x) = +\infty$. Assume furthermore, that there is an entire function $g(z) = \sum_{j=0}^{\infty} a_j z^j$ such that*

$$f'(z) = g(z)\,\hat{g}(z) \qquad \textit{where} \quad \hat{g}(z) = \sum_{j=0}^{\infty} \bar{a}_j z^j \tag{4.25}$$

and $\bar{a}_j$ is the conjugate of a_j, and that there are constants

$$A > 0, \quad \theta > 0, \quad s > 1 \qquad \textit{and} \qquad r_0 > 0$$

such that

$$M_g(r, s) \leqslant A(\|f\|_r)^\theta \qquad \textit{for} \quad r \geqslant r_0. \tag{4.26}$$

Then f has geometric convergence.

Proof. If one defines

$$h_1(z) = \tfrac{1}{2}(g(z) + \hat{g}(z))$$

and

$$h_2(z) = (1/2i)(g(z) - \hat{g}(z))$$

then

$$f'(z) = h_1^2(z) + h_2^2(z),$$

and h_1 and h_2 are real valued on $[0, +\infty)$.

It is now easy to see that h_1 and h_2 satisfy the conditions of Theorem 4.1. Hence, by Theorem 4.1 f has geometric convergence.

We remark that there are sufficient conditions in the literature for a function to satisfy (4.25) (cf. [3]).

5. Remarks and Conclusions

Example 4.1 shows that it is possible for a function with geometric convergence to oscillate somewhat. On the other hand, Theorem 3.1 shows that such functions cannot oscillate too much.

It appears that the complete characterization of functions f with geometric convergence will have to involve the rate of growth of

$$M_f(r)/m_f(r) \qquad \text{as} \quad r \to +\infty$$

where $M_f(r) = \|f\|_r$ and $m_f(r) = \inf_{x \geq r} |f(x)|$, as well as the necessary conditions in Theorem 2.1.

Another interesting question is whether f has geometric convergence if it satisfies the necessary conditions in Theorem 2.1 and is increasing on $[r_1, +\infty)$ for some $r_1 \geq r_0$.

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